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General solution of the problem of design of laminated plates possessing the given stiffnesses

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Abstract

The paper deals with the problem of design laminated plates possessing the given stiffnesses. The following two design problems are considered: (1) the continuous design problem (when one can use materials with any elastic characteristics to manufacture the plate); (2) the discrete design problem (when one can use a finite set of materials). It is known that design problems are closely related with convex analysis problems. It is shown that the laminated plate design problems are related with the convex combinations problem (CCP).

Using the CCP technique, one can analyze the laminated plate design problems in detail. This paper is concentrated on the general solution of the design problem (i.e. the set of all solutions of the problem). The general solutions are constructed for both the continuous and the discrete design problems. The methods developed in the paper can be used to solve the design problem numerically. © 2001 Elsevier Science Ltd. All rights reserved.

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1. Introduction

Laminated plates are widely used in the modern structures. In many cases the plate can be effectively used if it has prescribed properties. It leads to design problem. In its engineering arrangement, this problem has the following form: the material characteristics of laminas and distribution of laminas must be specified so that a plate formed from them will have a given set of stiffnesses.

Numerous investigators analyzed the design problem for laminated plate. It is impossible to give a full list of the papers devoted to this problem here (see the book by Zafer et al. (1999) giving an introduction into the modern state of the problem and references). The most part of the papers was devoted to the optimal design problem formulated as follows: Find a design giving to the plate an optimal property (for example, the minimal weight or the maximal stiffness). The design problem considered in this paper is formulated as follows: Find a design giving to the plate the required property (in particular, the optimal one).

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Generally, the design problem may not have a unique solution. This means that a plate, which possesses a given set of stiffnesses, can be designed in many ways. This paper is concentrated on description of all possible designs.

The relationship between the overall properties of an inhomogeneous plate and the properties of the materials forming the plate is established by the homogenization theory (see Kalamkarov and Kolpakov (1997), Caillerie (1984), Ciarlet (1990), Destuynder (1986), Kohn and Vogelius (1984) and Panasenko and Reztsov (1987)). In the general case this relationship is very complex. For a laminated plate the local material characteristics depend on only one spatial variable. This makes it possible to obtain explicit formulas for calculation of the stiffnesses of laminated plates and to reduce the design problem to an integral equation of first order.

For laminated solids the design problem was solved by Kolpakov and Kolpakova (1991, 1995) using the convex combinations problem (CCP) technique. The laminated plate design problem involves the coordinate across the plate. It changes the problem drastically. For the case when there are no restrictions on the constitutive materials (materials with any characteristics are available to manufacture the plate) the plate design problem can be analyzed on the basis of Pontraygin's extremal principle (see Kolpakov (1989)) and the set of possible values of stiffnesses can be described. In the present paper this problem is analyzed on the basis of the CCP technique. This approach provides information not only about the stiffnesses but also about the general solutions of the problem.

In many cases one can use a finite set of materials to manufacture the plate. It leads to the mathematical formulation of the problem in the form of a discrete problem (some analog of an integer-programming problem). The problem arising is called below the discrete CCP. A method for solution of the discrete CCP is developed. The method can be used to solve the discrete design problem numerically.

The paper is organized as follows. In Section 2 the statement of the problem is given. In Section 3 some results concerning the CCP are presented. Sections 4 and 5 deal with the continuous design problem. Section 6 is devoted to plates of symmetrical structure. In Section 7 the discrete design problem is formulated and reduced to a discrete CCP. In Section 8 a method solution of the discrete CCP is described. Sections 9 and 10 are devoted to the case when the stiffnesses are given of non-exactly or a combination of stiffnesses is given.

2. The statement of the problem

Let us consider a laminated plate formed from layers of homogeneous isotropic materials parallel to the Ox_1x_2 plane (Fig. 1).

Denote by $z = x_3/h$ the undimensional coordinate transverse to plate (h means the total thickness of the plate). The material characteristics of the plate (Young's modulus $E(z)$ and Poisson ratio $\nu(z)$) are functions of the variable z. The stiffnesses of the plate (the in-plane stiffnesses S^0_{ijkl}, the coupling stiffnesses S^1_{ijkl} and the bending stiffnesses S^2_{ijkl}, $ijkl = 1, 2$) are related with the variable z and the functions $E(z)$, $\nu(z)$ by the following formulas (see e.g. Kalamkarov and Kolpakov (1997)): ($\mu = 0, 1, 2;\ i, j, k, l = 1, 2$)

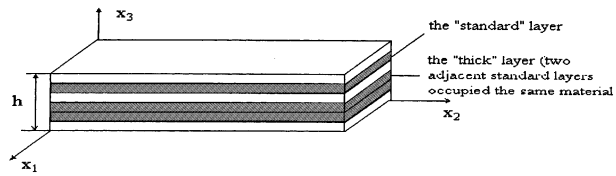

Fig. 1. The laminated plate.

$$h^{\mu+1}\int_{-1/2}^{1/2} E(z)z^{\mu}/(1-v^2(z))\,\mathrm{d}z = S^{\mu}_{iiii} \quad \text{for } ijkl = iiii \tag{2.1}$$

$$h^{\mu+1}\int_{-1/2}^{1/2} E(z)z^{\mu}/(1+v(z))\,\mathrm{d}z = S^{\mu}_{1212} \quad \text{for } ijkl = 1212, 2121 \tag{2.2}$$

$$h^{\mu+1}\int_{-1/2}^{1/2} E(z)v(z)z^{\mu}/(1-v^2(z))\,\mathrm{d}z = S^{\mu}_{1122} \quad \text{for } ijkl = 1122, 2211 \tag{2.3}$$

The other stiffnesses are equal to zero.

With regard to the formulas (2.1)–(2.3) the design problem may be formulated as follows: Solve the Eqs. (2.1)–(2.3) for $(v, i, j, k, l) \in S$ with respect to the functions $E(z)$ and $v(z)$. Here S means the indices of the given stiffnesses (all or only some of the stiffnesses may be included in the set S of the given stiffnesses).

Let us consider the case $v = \text{const}$. In this case the problems (2.1)–(2.3) may be reduced to the problem of the form

$$\int_{-1/2}^{1/2} E(z)z^{\mu}\,\mathrm{d}z = S^{\mu}, \quad \mu = 0, 1, 2 \tag{2.4}$$

with respect to the unique function $E(z)$. The quantities S^{μ} in (2.4) are expressed through the given stiffnesses S^{μ}_{ijkl}.

Note 2.1: If the number of layers is large (the thickness of layer is small compared with the thickness of plate) the stiffnesses may be calculated using the classical formulas ($S^2_{1111} = Eh^3/(1-v^2)$ and so on) where the Young's modulus E and the Poisson's ratio v are calculated in accordance with the homogenization procedure for 3-D body (see for details Kolpakov (1982)). If the number of layers is not large this method can lead to incorrect results.

Note 2.2: If $v \neq \text{const}$, we can solve the problems (2.1)–(2.3) with respect the functions $I_1(z) = E(z)/(1-v^2(z))$, $I_2(z) = E(z)/(1+v(z))$, $I_3(z) = E(z)v(z)/(1-v^2(z))$ and then construct the general solution, (see Kolpakov (2000)).

3. The convex combinations problem

Analyzing the design problem, we will use the methods based on the CCP technique developed by Kolpakov and Kolpakova (1991, 1995). Given here some results from Kolpakov and Kolpakova (1991, 1995) (see also Kalamkarov and Kolpakov (1997)), which will be used in the next sections.

Let $\{\mathbf{v}_i,\ i = 1, \dots, m\}$, $\mathbf{v} \in \mathbb{R}^k$ be given vectors. Consider the following problem with respect to the real numbers $\{x_i,\ i = 1, \dots, m\}$:

$$\sum_{i=1}^{m} \mathbf{v}_i x_i = \mathbf{v} \tag{3.1}$$

$$\sum_{i=1}^{m} x_i = 1, \quad 0 \leqslant x_i \leqslant 1 \tag{3.2}$$

It is the so-called CCP. It is known (see Kolpakov and Kolpakova (1991)) that the set $\Lambda(v)$ of all the solutions of the problems (3.1) and (3.2) (the so-called general solution) is given by the following formula

$$x_i = \sum_{\gamma=1}^{M} P_{i\gamma}\lambda_\gamma, \quad i = 1, \ldots, m \tag{3.3}$$

where $\{\mathbf{P}_\gamma = (P_{1\gamma}, \ldots, P_{m\gamma}),\ \gamma = 1, \ldots, M\}$ is a finite set of solutions of CCP; M is the total number of these solutions; and $\{\lambda_\gamma,\ \gamma = 1, \ldots, M\}$ are arbitrary real numbers satisfying the conditions

$$\sum_{\gamma=1}^{M} \lambda_\gamma = 1, \quad 0 \leqslant \lambda_\gamma \leqslant 1 \tag{3.4}$$

or, that is the same, the set $\Lambda(\mathbf{v})$ can be represented as

$$\Lambda(\mathbf{v}) = \mathrm{conv}\{\mathbf{P}_\gamma,\ \gamma = 1, \ldots, M\} \tag{3.5}$$

'conv' means the 'convex hull' (Rockafellar (1970)).

Note 3.1: An algorithm for calculation the solutions $\{\mathbf{P}_\gamma,\ \gamma = 1, \ldots, M\}$ was developed by Kolpakov and Kolpakova (1991, 1995) (see also Kalamkarov and Kolpakov (1997)).

4. Design of laminated plates with the required bending stiffness

The bending stiffness S^2_{1111} of the laminated plate is calculated as (see Eq. (2.1))

$$S^2_{1111} = h^3/12(1 - \nu^2) \int_{-1/2}^{1/2} E(z) z^2 \, \mathrm{d}z \tag{4.1}$$

The design problem is formulated as follows: Find all distributions of Young's modulus $E(z)$ which satisfy Eq. (4.1) with the given S^2_{1111}.

Eq. (4.1) is not a CCP. Nevertheless, we can transform it into a CCP. Introduce the new measure μ determined by the equation: $\mathrm{d}\mu(z) = z^2\,\mathrm{d}z$ $(\mu(z) = (z^3/3) + (1/8))$. Then Eq. (4.1) becomes

$$12 S^2_{1111}(1 - \nu^2)/h^3 = \int_{2/24}^{4/24} E(z)\, \mathrm{d}\mu(z) \tag{4.2}$$

For a plate formed of m homogeneous materials we can rewrite Eq. (4.2) as the following CCP

$$\sum_{i=1}^{m} E_i \mu_i = C, \quad \sum_{i=1}^{m} \mu_i = 1/12, \quad \mu_i \geqslant 0 \tag{4.3}$$

where $C = 12 S^2_{1111}(1 - \nu^2)/h^3$. Introducing the variables $x_i = 12\mu_i$, we can write the problem (4.3) in the forms (3.1) and (3.2).

In accordance with Eqs. (3.3) and (3.4) we can write down the general solution of the problem (4.3) in the form

$$\mu_i = 1/12 \sum_{\gamma=1}^{M} P_{i\gamma}\lambda_\gamma \quad \text{for any} \quad \{\lambda_\gamma \geqslant 0\} \text{ such that } \sum_{\gamma=1}^{M} \lambda_\gamma = 1 \tag{4.4}$$

Consider the domain $M_p = \{\mu\colon E(\mu) = E_p\}$ corresponding to the pth material $(p = 1, \ldots, m)$. Note that M_p is not the domain occupied by the pth material and measure μ_p of M_p is not the volume ratio of the pth material. In order to find the design we must return to the initial coordinate z. This procedure is illustrated in Fig. 2 for the case of a plate made of two materials. The curve in Fig. 2 is the graph of the function $\mu(z) = (z^3/3) + (1/8)$. The solutions presented in Fig. 2(b) and (c) are corresponding to the maximal and the minimal volume ratio of phase in composite.

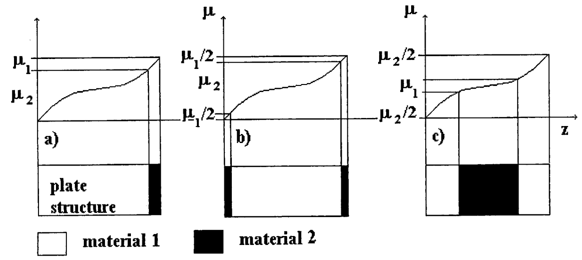

Fig. 2. Two-material designs of laminated plate with the same bending stiffness according to different distributions of the sets M_1, M_2 (having measures μ_1 and μ_2, respectively). The sets M_1 and M_2 are distributed in $O\mu$ axis. The plate structure arises in Oz axis.

5. Design of laminated plate with the required in-plane and bending stiffnesses

The in-plane stiffness S^0_{1111} of the laminated plate is calculated as (see Eq. (2.1))

$$S^0_{1111} = h/(1-\nu^2)\int_{-1/2}^{1/2} E(z)\,\mathrm{d}z \tag{5.1}$$

The equality (5.1), be considered with respect to the function $E(z)$ taking m possible values, is a CCP. Its solution is given by the formula (2.3).

Introduce the set $L_p = \{z\colon E(z) = E_p\}$ – domain occupied by the pth material ($p = 1, \ldots, m$). The measure x_p of L_p is the volume ratio of the pth material.

Consider the design problem when only two materials are available. In this case the variable x_i takes values x_1 and x_2; and the variable μ_i takes values μ_1 and μ_2. The condition that the plate has the in-plane stiffness S^0_{1111} and the bending stiffness S^2_{1111} can be written in the form (see Sections 3 and 4)

$$\sum_{r=1}^{K} L_r = x_1, \qquad \sum_{r=1}^{K} M_r = \mu_1 \tag{5.2}$$

where x_1 is any solution of CCP corresponding to Eq. (5.1) and μ_1 is any value given by Eq. (4.3).

Here $\{M_r\}$ mean the lengths of the intervals forming the set M_1 (see Section 4) and $\{L_r\}$ means the lengths of the intervals $[a_{2r}, a_{2r+1}]$ ($r = 1, \ldots, K$) forming the set L_1.

In addition to Eq. (5.2) we have the following condition

$$M_r = \int_{a_{2r}}^{a_{2r+1}} z^2\,\mathrm{d}z \tag{5.3}$$

Systems (5.2) and (5.3) are solvable only if $S^2_{1111} \in [L_{\min}, L_{\max}]$, where $L_{\max}$ is determined in Fig. 2(b) and $L_{\min}$ is determined in Fig. 2(c).

5.1. The partial solutions of the design problems (5.2) and (5.3)

A partial solution of the problems (5.2) and (5.3) can be constructed in the following way. Let us introduce an interval $[a, b]$ whose length M_1 is equal to x_1 (it is the interval occupied by the first material). We will obtain a partial solution of the problem if we find a and b such that the following equation is satisfied

$$\int_a^b z^2\,\mathrm{d}z = \mu_1 \text{ and } b - a = x_1 \tag{5.4}$$

Eq. (5.4) can be written as

$$F(x) = \mu_1, \quad 0 \leqslant x \leqslant x_1 + x \leqslant 1 \tag{5.5}$$

where

$$F(x) = \int_x^{x_1+x} z^2 \, dz = \mu_1$$

This equation with respect to the variable x can be solved numerically (Minoux (1989)).

One can consider the more general case when the first material occupies K layers (the thicknesses and the positions of the layers are unknown). In this case we come to the following equation

$$F(\mathbf{x}) = \sum_{r=1}^{K} \int_{a_{2r}}^{a_{2r}+x_r} z^2 \, dz = \mu_1, \quad a_{2r} + x_r \leqslant a_{2(r+1)} \tag{5.6}$$

which is an algebraic equation for a function of $2K$ variables. It can be solved numerically (Minoux (1989)).

Note 5.1: The number K of the layers is arbitrary. It is interesting to know the minimal number of layers sufficient to design the plate.

5.2. Solution corresponding to the minimal number of layers

Consider the following problem: Indicating the minimal number of layers that allow designing a plate with every possible value of stiffnesses. A problem of such kind (different from the problem considered here) was analyzed by Kolpakov (1989).

In the case under consideration (remember that we consider plates made of two materials) solution is the following: the first material must be distributed among *two* (not more) layers.

Really, solutions corresponding to $L_{\min}$ and $L_{\max}$ are not more than two layers solutions; see Fig. 2(b) and (c) (here we say about the layers occupied by the first material). The problems (4.1) and (5.1) continuously depend on the function $E(z)$. Then, continuously transforming the function $E(z)$ from the first solution to the second solution (obviously, it is possible) one obtains solution of the problems (4.1) and (5.1) for every $S^2_{1111} \in [L_{\min}, L_{\max}]$.

Reminding about the second material forming the plate, we conclude that the total number of layers is equal to five.

Thus, solution of the problem (5.6) with $K = 2$ provides a design for every possible values of in-plane and bending stiffnesses. The problem (5.6) with $\mathbf{x} \in \mathbb{R}^2$ may be solved numerically (Minoux (1989)).

Note 5.2: The results obtained in Sections 4 and 5 may be useful in design of plates of symmetric (with respect to the plane $z = 0$) structure. For the non-symmetrical plates it is necessary to take into account the coupling stiffnesses.

6. The plates of symmetrical structure

Let us consider a plate with balanced placement of laminae about Ox_1x_2 plane (called also plates of symmetrical structure). For such a plate all the out-off-plane stiffnesses $S^1_{ijkl} = 0$ and, consequently, Eqs. (2.1)–(2.3) with $v = 0, 2$ (or Eqs. (4.1) and (5.1)) represent the general form of the problem.

Consider the case when the stiffness S^0_{1111} is fixed. Using the designs presented in Fig. 2(b) and (c) we can find that the bending stiffness S^2_{11111} can take any value between the minimal value

$$S_{\min} = \frac{h^3}{12(1-v^2)} \left[\frac{\left(S^0_{1111}/h - E_2\right)^3}{\left(E_1 - E_2\right)^2} + E_2 \right]$$

and the maximal value

$$S_{\max} = \frac{h^3}{12(1-v^2)}\left[-\frac{\left(S^0_{1111}/h - E_1\right)^3}{\left(E_1 - E_2\right)^2} + E_1\right]$$

Here $E_1 > E_2$ are Young's modulus, v is the Poisson's ratio (it is assumed the same for both the materials), h is the thickness of the plate.

We can obtain designs of the plate using the procedure described in Section 4. In order to obtain a symmetric design we must distribute the sets M_p in $O\mu$ axis symmetrically with respect the line $\mu = 1/2$.

7. The discrete design problem

In the following sections we consider the case when a finite number of materials (indexed by the numbers $1, \ldots, n$) is available to manufacture the plate. It means that the function $E(z)$ takes values in a finite set $Z_n = \{E_1, \ldots, E_n\}$.

Let us divide the segment $[-1/2, 1/2]$ (the plate thickness in undimensional variable z) into m intervals $[-1/2 + (i-1)/m, -1/2 + i/m)$. It means that we divide the plate into m laminae of thickness $1/m$. The function $E(z)$ is constant over the interval $[-1/2 + (i-1)/m, -1/2 + i/m)$.

Note 7.1: If the function $E(z)$ takes the same value in adjacent intervals it means that the material occupies a "thick" layer.

Denote

$$d_{1i} = \delta^{-1}\int_{-1/2+(i-1)m}^{-1/2+i/m} z\,\mathrm{d}z, \quad d_{2i} = \delta^{-1}\int_{-1/2+(i-1)m}^{-1/2+i/m} z^2\,\mathrm{d}z, \qquad \delta = 1/m$$

Then the design problem (2.4) may be written in the form

$$\sum_{i=1}^{m} E_i\delta = S^0, \quad \sum_{i=1}^{m} E_i d_{1i}\delta = S^1, \quad \sum_{i=1}^{m} E_i d_{2i}\delta = S^2 \tag{7.1}$$

where $\{E_i \in Z_n,\ i = 1, \ldots, n\}$ are unknowns. The unknowns $E_i > 0$ in accordance with the nature of Young's modulus.

The problem (7.1) can be written as

$$\sum_{i=1}^{m} x_i = 1, \quad x_i \in Z_n, \qquad \sum_{i=1}^{m} x_i\mathbf{v}_i = \mathbf{v} \tag{7.2}$$

where $x_i = E_i\delta/S^0$, $\mathbf{v}_i = (d_{1i}, d_{2i}),\ i = 1, \ldots, m$; $\mathbf{v} = (S^1/S^0, S^2/S^0)$.

8. The discrete convex combinations problem

We consider the problem (7.2) in the general form. Let $Z_n \subset [0, 1]$ be a finite set (consisting of n numbers); $\{\mathbf{v}_i,\ i = 1, \ldots, m\}$, $\mathbf{v} \in \mathbb{R}^k$ be the given vectors. Consider the following problem with respect to the numbers $\{x_i,\ i = 1, \ldots, m\}$:

$$\sum_{i=1}^{m} \mathbf{v}_i x_i = \mathbf{v} \tag{8.1}$$

$$\sum_{i=1}^{m} x_i = 1 \tag{8.2}$$

$$x_i \in Z_n, \quad i = 1, \dots, m \tag{8.3}$$

The problems (8.1)–(8.3) will be called a discrete CCP.

In this section we construct the general solution of the problems (8.1)–(8.3) – the set $\Lambda(\mathbf{v})$ of all the solutions of the problems (8.1)–(8.3).

Omitting the condition of discreteness (8.3) in problems (8.1)–(8.3), we obtain a continuous CCP (3.1) and (3.2). The general solution $\Lambda(\mathbf{v})$ of the continuous CCP (3.1) and (3.2) is given by the formulas (3.3) and (3.4).

Then to solve the discrete problems (8.1)–(8.3) it is sufficient to select in $\Lambda(\mathbf{v})$ the vectors whose coordinates satisfy the condition (8.3).

Present an algorithm performing this selection. Problems (3.3) and (3.4) may be considered as CCP (with respect to unknowns λ_γ). We must find $x_1, \dots, x_m$ for whose the CCP (3.3) and (3.4) is solvable and which belong to the set Z_n. We use the following property of CCP (see Kolpakov and Kolpakova (1991, 1995)): If the first $(i-1)$ equations in (3.3) with the conditions (3.4) are satisfied then the ith equation in Eq. (3.3) is solvable if and only if

$$x_i \in [\min_i, \max_i] \tag{8.4}$$

From Eq. (8.4) we obtain the following necessary and sufficient condition solvability of the discrete CCP:

$$Z(i, \mathbf{x}(i-1)) = Z_n \cap [\min_i, \max_i] \neq \varnothing \quad \text{for any } i = 1, \dots, m \tag{8.5}$$

Now we describe an iterative algorithm constructing all the vectors $\mathbf{x}$ satisfying (8.5).

- In the first step we take an arbitrary start point $T(0) = \{x_0\}$ (the root of a tree T).
- In the $(i-1)$th step we have a set $T(i-1)$ of the points $\{\mathbf{x}(i-1)\} = \{(x_0, \dots, x_{i-1})\colon x_1, \dots, x_{i-1} \in Z_n\}$ for which the first $(i-1)$ equations in Eq. (3.3) with the condition (3.4) are solvable. Let us calculate the intervals $Z(i, \mathbf{x}(i-1))$ corresponding to all the vectors $\mathbf{x}(i-1) \in T(i-1)$. After that let us construct the set $T(i)$ of all the vectors of the form $\mathbf{x}(i) = (x_0, \dots, x_{i-1}, x_i)$ where $(x_0, \dots, x_{i-1}) = \mathbf{x}(i-1) \in T(i-1)$ and $x_i \in Z_n \cap Z(i, \mathbf{x}(i-1))$ (if $Z_n \cap Z(i, \mathbf{x}(i-1)) \neq \varnothing$).
- If $Z_n \cap Z(i, \mathbf{x}(i-1)) = \varnothing$ for every $\mathbf{x}(i-1) \in T(i-1)$ then stop (the discrete CCP is not solvable).
- If $i = m$ then stop (the discrete CCP is solvable).

The tree $T(m)$ has the following property. If the mth level of the tree $T(m)$ is not empty then the discrete CCP (8.1)–(8.3) is solvable (otherwise it has no solution) and any vector $\mathbf{x}(m) \in T(m)$ (a branch of the tree) is solution of the discrete CCP (8.1)–(8.3). On the other hand any solution $\mathbf{x}$ of the discrete CCPs (8.1)–(8.3) is a branch of the tree $T(m)$. It means that the set of all the branches $(x_1, \dots, x_{i-1})$ (is not taken into account) of the tree $T(m)$ is the general solution $\Delta(\mathbf{v})$ of the discrete CCP.

8.1. Numerical algorithms

Transition from CCPs (3.1) and (3.2) to CCPs (3.3) and (3.4) – *calculation of the vectors* $\{\mathbf{P}_\gamma,\ \gamma = 1, \dots, M\}$*:* The vectors $\{\mathbf{P}_\gamma,\ \gamma = 1, \dots, m\}$ may be calculated on the base of the convolution algorithm presented in Kolpakov and Kolpakova (1991).

Calculation of the intervals $[\min_i, \max_i]$: In order to calculate the interval $[\min_i, \max_i]$ the simplex method can be applied. Using this method, we consider at the $(i-1)$th step the first $i-1$ equations in (3.3) and the condition (3.4) as a restrictions and introduce the coast function $L(\lambda)$ corresponding to the ith equation in Eq. (3.3) in the following way

$$L(\lambda) = \sum_{\gamma=1}^{M} P_{i\gamma}\lambda_{\gamma}$$

After that we find $\min_i$ ($\max_i$) solving the problem

$$L(\lambda) \to \min(\max)$$

with the restrictions described above.

Advantage of the method based on the simplex method is that this method does not generate large data.

The tree $T(m)$: The tree can be realized on the base of any known data structure.

8.2. Numerical calculations

The results of numerical calculations are presented in Kolpakov (2000).

9. The design problem for the stiffnesses given not exactly

As was noted by Kolpakov (2000) often there exist the designs, which do not satisfy the equations in Eq. (7.2) exactly but satisfy these equations approximately. It is clear that these solutions may be suitable for the practice. We write the design problem giving these solutions.

Let us consider the problem (7.2) with the condition that the quantities S^{μ}, $\mu = 0, 1, 2$ belong to intervals $[S^{\mu} - \delta S^{\mu}, S^{\mu} + \delta S^{\mu}]$, $n = 0, 1, 2$, where δS^{μ}, $n = 0, 1, 2$ are the allowed derivations of these quantities from the required values S^{μ}, $n = 0, 1, 2$.

It means that we can write Eq. (7.1) in the form

$$S^0 - \delta S^0 \leqslant \sum_{i=1}^{m} E_i\delta \leqslant S^0 + \delta S^0, \qquad S^1 - \delta S^1 \leqslant \sum_{i=1}^{m} E_i d_{1i}\delta \leqslant S^1 + \delta S^1,$$
$$S^2 - \delta S^2 \leqslant \sum_{i=1}^{m} E_i d_{2i}\delta \leqslant S^2 + \delta S^2 \tag{9.1}$$

Introducing the additional variables $x_{m+1}, \ldots, x_{m+5}$, we can write Eq. (9.1) in the form:

$$\begin{aligned}
&\sum_{i=1}^{m} x_i = 1, \quad x_i \geqslant 0, \\
&\sum_{i=1}^{m} x_i - x_{m+1} = 1 + \delta S^0/(S^0 - \delta S^0), \\
&\sum_{i=1}^{m} d_{1i}x_i + x_{m+2} = v_1, \qquad \sum_{i=1}^{m} d_{1i}x_i - x_{m+3} = v_1 + \delta S^1/(S^1 - \delta S^1), \\
&\sum_{i=1}^{m} d_{2i}x_i + x_{m+4} = v_1, \qquad \sum_{i=1}^{m} d_{2i}x_i - x_{m+5} = v_2 + \delta S^2/(S^2 - \delta S^2),
\end{aligned} \tag{9.2}$$

where d_{1i}, d_{2i} are determined in Section 6 and $x_i = E_i\delta/(S^0 - \delta S^0)$, $v_1 = (S^1 + \delta S^1)/(S^1 - \delta S^1)$, $v_2 = (S^2 + \delta S^2)/(S^2 - \delta S^2)$.

The problem (9.2) can be transformed into CCP. Note that the problem (9.2) itself is not a CCP because the first equation in Eq. (9.2) does not include all the variables.

10. The case when a combination of stiffnesses is given

Consider the case when we want to give a required value not to the stiffnesses themselves but to some combination of the stiffnesses. The design criterion takes the form

$$D(S^{\mu}_{ijkl}) = D_0 \tag{10.1}$$

where D is a known function.

Denote by L the set solutions of Eq. (10.1) with respect to S^{μ}_{ijkl}. Note, that S^{μ}_{ijkl} are not the design variables. Then the design problem is reduced to the following

$$S^{\mu}_{ijkl} = \mathbf{l}, \quad \mathbf{l} \in L \tag{10.2}$$

For any $\mathbf{l}$ problem (10.2) can be solved as above and write the general solution Λ of the problem (10.1) in the form $\Lambda = \bigcup_{\mathbf{l}\in L} \Lambda(\mathbf{l})$, where $\Lambda(\mathbf{l})$ means solution of Eq. (10.2) for a fixed $\mathbf{l} \in L$.

An example. The numerical values of the coupling and bending stiffnesses depend on choose of the coordinate system. Consider the stiffnesses S^{μ}_{ijkl} and $S^{\mu}_{ijkl}(h)$ and quantities S^{μ} and $S^{\mu}(h)$ (2.4) ($\mu = 0, 1, 2$) computed with respect to coordinate systems z and $z + h$ (h is an arbitrary number), correspondingly. The quantities S^{μ} and $S^{\mu}(h)$ are related as follows (see Kolpakov (1999)):

$$S^0(h) = S^0, \quad S^1(h) = S^1 + hS^0, \quad S^2(h) = S^2 + 2hS^1 + h^2S^0 \tag{10.3}$$

Using Eq. (10.3), we can form the following two invariants (functions depending on S^{μ}_{ijkl}, but not depending on h):

$$D_1(S^{\mu}_{ijkl}) = S^0(h)/(1 - v^2), \quad D_2(S^{\mu}_{ijkl}) = (S^2(h) - (S^1(h))^2/S^0(h))/(1 - v^2) \tag{10.4}$$

In Eq. (10.4) v means the Poisson's ratio.

The invariants (10.4) represent *the physical* in-plane and bending stiffness. In particular, $D_2(S^{\mu}_{ijkl})$ is the bending stiffness computed in the coordinate system such that $S^1(h) = 0$.

In the case under consideration the system (10.2) takes the form

$$\int_{-1/2}^{1/2} E(z)\,\mathrm{d}z = S^0, \quad \int_{-1/2}^{1/2} E(z)z\,\mathrm{d}z = l, \quad \int_{-1/2}^{1/2} E(z)z^2\,\mathrm{d}z = S^2 - l^2/S^0 \tag{10.5}$$

where l is an arbitrary number (a parameter). Analyzing the second integral in Eq. (10.5), we find that $l \in L = \left[-\frac{E_{\max} - E_{\min}}{8}, \frac{E_{\max} - E_{\min}}{8}\right]$, where $E_{\max}$ and $E_{\min}$ mean the maximal and minimal values of $E(z)$, correspondingly.

The method developed in Sections 4–6 is suitable to solve the problem (10.5) with $l = 0$ and the method developed in Sections 7 and 8 is suitable to solve the problem (10.5) with arbitrary l.

11. Summary

It was shown that the laminated plate design problem is related with the CCP. To write the CCP corresponding to the plate design problem we proposed to use Young's modulus as coefficients of the convex combinations.

We considered the continuous design problem (when one can use materials with any elastic characteristics to manufacture the plate) and the discrete design problem (when one can use a finite set of materials).

Using the CCP technique, we analyzed the continuous design problem in details. In particular we described the general solution (the set of all solutions) of the continuous design problem.

We presented a method solution the discrete CCP. This method can be used to solve the discrete design problem numerically.

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